

Spectral action beyond the weak-field approximation

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Abstract

The spectral action for a non-compact commutative spectral triple is computed covariantly in a gauge perturbation up to order 2 in full generality. In the ultraviolet regime, $p \rightarrow \infty$, the action decays as $1/p^4$ in any even dimension.

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1 Introduction

Recent advances [12, 14] in explaining some key features of gravity and Standard Model through the spectral action of noncommutative geometry brought this subject to a focus of interest in theoretical physics. In noncommutative geometry, all information is encoded in a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where \mathcal{A} is an algebra acting on a Hilbert space \mathcal{H} and \mathcal{D} is a selfadjoint operator on \mathcal{H} which plays the role of a Dirac operator [13, 14, 19]. In this approach, the action is the so-called spectral action introduced by Chamseddine and Connes [9–11]

$$S(\mathcal{D}, \Lambda, f) = \text{Tr} (f(\mathcal{D}^2/\Lambda^2)) \quad (1)$$

and defined for $\Lambda \in \mathbb{R}^+$ which plays the role of a cut-off (and needed to make \mathcal{D}/Λ dimensionless) and for a function f such that, of course, $f(\mathcal{D}^2/\Lambda^2)$ is trace-class. In general, one chooses $f \geq 0$ since the action $\text{Tr} (f(\mathcal{D}^2/\Lambda^2)) \geq 0$ will have the correct sign for an Euclidean action. This action is the appropriate one in the framework of noncommutative geometry to reproduce several physical situations like the Einstein-Hilbert action in gravitation or the Yang-Mills-Higgs action in the standard model of particle physics [14], and the positivity of the function f implies positivity of actions for gravity, Yang-Mills or Higgs couplings, and the Higgs mass term is negative.

Till the end of this Section we shall present a non-technical summary of our results to give a more physics-oriented reader a chance to appreciate them without going through the mathematics of the rest of this paper.

Let $M = \mathbb{R}^{2m}$ be an even dimensional real plane, $d = 2m \geq 2$, endowed with a spin structure given by the spinor bundle $S = \mathbb{C}^{2^m}$. We denote by \mathcal{D} the free Dirac operator and by \mathcal{D}_A the standard Dirac operator with a gauge connection A acting on the Hilbert space $\mathcal{H} := L^2(M, S)$. We will use the notations and conventions from [25, eq. (3.26)], namely in local coordinates

$$\mathcal{D} := i\gamma^\mu \partial_\mu, \quad \mathcal{D}_A := i\gamma^\mu \nabla_\mu := i\gamma^\mu (\partial_\mu + A_\mu) \quad (2)$$

where $A_\mu \in \Gamma(M, \text{End}(S))$ is taken in some representation of complex dimension N of the gauge group. We assume that A_μ with field strength $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ satisfies for a given $\rho > d$,

$$|A_\mu(x)| \leq c(1 + |x|)^{-\rho}, \quad \partial^\beta A_\mu(x) = \mathcal{O}(1), \quad (3)$$

for all $\beta \in \mathbb{N}^d$ with $|\beta| \leq m$. This hypothesis will be justified in Section 2. Since the manifold $M = \mathbb{R}^{2m}$ is non-compact, $f(\mathcal{D}^2)$ is never trace-class, so, in order to get rid of trivial volume divergence, we modify (1) as

$$S(\mathcal{D}_A, \Lambda, f) := \text{Tr} (f(\mathcal{D}_A^2/\Lambda^2) - f(\mathcal{D}^2/\Lambda^2)). \quad (4)$$

It is known that $(\mathcal{S}(\mathbb{R}^d), \mathcal{H}, \mathcal{D})$ is a spectral triple with the non-unital algebra of Schwartz functions, see definition in [8, 16] and [16] for a proof. Other algebras are possible, like smooth functions on \mathbb{R}^d with arbitrary partial derivatives bounded and integrable. Moreover, (4) can be seen as a variation of the spectral actions when a one-form is turned on, which also makes sense for non-unital spectral triples. Actually, the two spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and $(\mathcal{A}, \mathcal{H}, \mathcal{D}_A)$ represent the same geometry, because obtained via Morita equivalence [14, 19]. In the heat kernel asymptotics when $\Lambda \rightarrow \infty$, the coefficient of Λ^0 will be given by $\zeta_{\mathcal{D}_A}(0) - \zeta_{\mathcal{D}}(0)$ which has been computed in [10] (in case \mathcal{A} has a unit) and the kernels of \mathcal{D}_A and \mathcal{D} will not appear. Unfortunately, this does not give access to a definition of spectral action for a generic non-compact spectral triple, but it avoids the use of an extra smearing function g which can regularize $\text{Tr} (g f(\mathcal{D}^2/\Lambda^2))$. Such g has been used for instance in [17, 18, 25].

The trace formula (11) gives $S(\mathcal{D}_A, \Lambda, f) = \int_{-\infty}^{\infty} \xi(\lambda) f'(\lambda) d\lambda$ for the spectral shift function $\xi(\lambda)$. Thus (4) makes sense for a large class \mathcal{C}_1 of functions f described in Section 2.

The spectral action is quite well studied in the framework of the *weak-field expansion*, which is constructed in the following way (see [14] for a detailed treatment). It is assumed now that f is a Laplace transform of a function φ :

$$f(z) = L\varphi(z) := \int_0^{\infty} dt e^{-tz} \varphi(t). \quad (5)$$

Then,

$$S(\mathcal{D}_A, \Lambda, f) = \int_0^{\infty} dt \tilde{K}(t/\Lambda^2, \mathcal{D}_A^2) \varphi(t), \quad (6)$$

where

$$\tilde{K}(s, \mathcal{D}_A^2) = \text{Tr} \left(e^{-s\mathcal{D}_A^2} - e^{-s\mathcal{D}^2} \right) \quad (7)$$

is the subtracted heat kernel, which has an asymptotic expansion as $s \rightarrow +0$

$$\tilde{K}(s, \mathcal{D}_A^2) \underset{s \downarrow 0}{\sim} \sum_{k=1}^{\infty} s^{-m+k} a_{2k}(\mathcal{D}_A^2). \quad (8)$$

Then

$$S(\mathcal{D}_A, \Lambda, f) \underset{s \downarrow 0}{\sim} \sum_{k=1}^{\infty} \Lambda^{2(m-k)} \varphi_{2k} a_{2k}(\mathcal{D}_A^2) \quad (9)$$

with

$$\varphi_{2k} = \int_0^{\infty} dt t^{-m+k} \varphi(t). \quad (10)$$

The heat kernel coefficients a_n are very well known at least in the commutative case [18, 25]. Let us assign canonical mass dimension 1 to A_μ and to the derivative. Each coefficient a_{2k} is an integral of a polynomial of the connection A_μ and its' derivatives of the total canonical dimension $2k$. The expansion (9) is valid therefore when the fields and their derivatives are small compared to Λ . Hence, this is a weak-field expansion.

The expansions (8) and (9) do not contain the term with a_0 , which cancels out between the \mathcal{D}_A and \mathcal{D} . This is, however, not a generic feature of spectral actions, but is rather a consequence of taking the same flat Euclidean metric on \mathbb{R}^d in both operators. In general, one has to allow for metric perturbations, see e.g. [14]. Let us consider the metric $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ where the fluctuations $h_{\mu\nu}$ over the unit metric are well localized (fall off sufficiently fast at infinity). Let us introduce this metric in \mathcal{D}_A through suitably defined γ -matrices and add corresponding connection terms. Let us leave \mathcal{D} as above. At this point we have to assume that the trace in (4) exists, though the case of metric fluctuations is not covered by the analysis of Section 2. Then the summation in (8) and (9) have to be extended to $k = 0$ with $a_0 \sim \int dx (\sqrt{\det g} - 1)$. Variation of $h_{\mu\nu}$ in the a_0 term in the spectral action produces the standard cosmological term in the equations of motion. Note, that the Einstein action with a cosmological term is always infinite on its' classical solutions on non-compact manifolds. Therefore, the necessity to subtract a contribution from some reference metric is well understood in General Relativity. For us here, it is only important that the a_0 term reappears in the spectral action with metric fluctuations and gives rise to the cosmological constant. This fact will be used for a physical interpretation of our results, see Remark 3.5.

If the spectral action is to be taken seriously, one should also study it beyond the weak-field approximation. In particular, the terms which depend quadratically on the field strength and contain arbitrary number of derivatives govern the ultraviolet behavior of propagators and are utterly important for quantization. One observes that, if one restricts himself to the second order in $F_{\mu\nu}$, the heat kernel coefficients have the form $a_{2k+4} \sim F_{\mu\nu} \Delta^k F_{\mu\nu}$ where $\Delta = -\partial_\mu \partial^\mu$ is the free

Laplacian. After a Fourier transform, this can be translated to $a_{2k+4} \sim \hat{F}_{\mu\nu}(-p) p^{2k} \hat{F}_{\mu\nu}(p)$. Then, it was suggested in [24] to pick up a function f such that the coefficients φ_{2k} vanish for $2k > 2m + N$, while $\varphi_{2m+N} \neq 0$ with a sufficiently large N . The spectral action restricted to first few coefficients in the heat kernel expansion displays the polynomial growth $\sim p^N$ and the corresponding propagators decay at large momenta. Power counting arguments show that the restricted Yang-Mills theory becomes super-renormalizable [24].

Using the Barvinsky-Vilkovisky approach of covariant perturbation theory [2, 3], the aim of this work is to compute the action (4) as a function of the field strength F of the connection A in full generality, then to control its ultraviolet regime. We restrict ourselves to the second order in F . To this order, we derive a remarkably simple formula for the spectral action for a large class of function f . Then, we show that the spectral action decays as $1/p^4$ in the ultraviolet asymptotics. Therefore, the propagator of the Yang-Mills field grows at large momenta, and the full Yang-Mills spectral action is not super-renormalizable in contrast to its expansion considered in [24].

This paper is organized as follows. Section 2 recalls few facts on the regularization deduced from Krein spectral shift function and conditions on f which guarantee that $f(\mathcal{D}_A^2/\Lambda^2) - f(\mathcal{D}^2/\Lambda^2)$ is a trace-class operator. Few known results where f is piecewise continuous are quoted. We use the G -pseudodifferential calculus to study some sufficient condition on the connection A that gives a trace-class resolvent perturbation.

In Section 3, we use the covariant perturbation expansion to compute the spectral action to the second order in $F(A)$ and its ultraviolet asymptotics. Our main result for the spectral action at the second order in F reads

$$S(\mathcal{D}_A, \Lambda, f)(F) = \frac{\Lambda^{d-4}}{(4\pi)^m} \int_M d^{2m}p \operatorname{tr} [\hat{F}^{\mu\nu}(-p) w_\Lambda(p^2) \hat{F}_{\mu\nu}(p)] + \mathcal{O}(F^3)$$

with the form-factor $w_\Lambda(p^2)$ given in Lemma 3.3 as

$$w_\Lambda(p^2) = (-1)^{m-1} 2^{m-2} \int_0^1 d\alpha \left[f^{[m-2]}(\alpha(1-\alpha)\Lambda^{-2}p^2) - \frac{2\Lambda^2}{p^2} \int_0^{\alpha(1-\alpha)\Lambda^{-2}p^2} ds_1 f^{[m-2]}(s_1) \right].$$

Here $f^{[n]}$ is the n th primitive of the function f . Dependence on the dimension $d = 2m$ of underlying space resides in the overall numerical factor and in the number of repeated integrations of f . The ultraviolet asymptotics (Theorem 3.4) of $w_\Lambda(p^2)$ depends on m through a numerical factor only.

Section 4 discusses how to relax hypothesis on the function f . In particular, if f is not assumed to be positive, one can construct a spectral action which decays faster as $1/p^4$. We also treat a step-function cut off.

Higher terms in the F -expansion are briefly considered in section 5.

Concluding remarks are given in section 6.

2 Non-compact spectral action and spectral shift function

The main purpose of this Section is to show that the trace in (4) exists under certain assumptions on f and A . We have start with a short tour into the theory of Krein spectral shift function.

2.1 Spectral shift function

The Krein spectral shift functions are widely used in scattering theory, see [21–23, 27–29]. For a pair of selfadjoint operators H_0, H on a Hilbert space \mathcal{H} , the spectral shift function $\xi(\lambda)$ is defined by the Lifshits trace formula

$$\operatorname{Tr} (f(H) - f(H_0)) = \int_{-\infty}^{\infty} d\lambda \xi(\lambda) f'(\lambda). \quad (11)$$

The idea behind is that, if $f(\lambda) = \chi_{]-\infty, \lambda[}$, then $\xi(\lambda) = -\text{“Tr”}(P(]-\infty, \lambda[) - P_0(]-\infty, \lambda[))$ where P, P_0 are the spectral projections of H, H_0 and “Tr” is some regularized trace, so $\xi(\lambda)$ appears to be a regularization of the difference of eigenvalue counting functions. Remark that $\xi(\lambda - 0)$ is equal to the index of the Fredholm pair $P(]-\infty, \lambda[)$ and $P_0(]-\infty, \lambda[)$ and coincides with the spectral flow of H and H_0 when λ is in the discrete spectrum of H_0 . These are different regularizations of $\text{Tr}(P(]-\infty, \lambda[) - P_0(]-\infty, \lambda[))$: the $\xi(\lambda)$ function is the regularization obtained by replacing the difference of spectral projections by $f(H) - f(H_0)$, where f is a smooth approximation of $\chi_{]-\infty, \lambda[}$, while the index is obtained by replacing Tr by index. These two regularizations does not coincide when λ is in the essential spectrum of H_0 since the function ξ is not always integer-valued as the index.

Assume that $V := H - H_0$ is in $\mathcal{L}^1(\mathcal{H})$ (trace-class operators). If $R(X)(z)$ is the resolvent of the operator X , the function $D(z) := \text{Det}(1 + V R(H_0)(z))$ is holomorphic and, moreover, satisfies $D^{-1}(z)D'(z) = \text{Tr}(R(H_0)(z) - R(H)(z))$. Defining

$$\xi(\lambda; H, H_0) := \pi^{-1} \lim_{\epsilon \downarrow 0} \arg D(\lambda + i\epsilon)$$

for almost all λ , we get the following:

$$\log D(z) = \int_{-\infty}^{\infty} d\lambda \xi(\lambda) (\lambda - z)^{-1} \text{ for } \Im(z) \neq 0, \int_{-\infty}^{\infty} d\lambda |\xi(\lambda)| \leq \|V\|_1 \text{ and } \text{Tr}(V) = \int_{-\infty}^{\infty} d\lambda \xi(\lambda).$$

Moreover, the trace formula holds at least for functions $f \in C_c^\infty(\mathbb{R})$ which are smooth with compact support.

Clearly, ξ deserves its name of spectral shift since when λ is an isolated eigenvalue for H and H_0 , with multiplicity n and n_0 , $\xi(\lambda + 0) - \xi(\lambda - 0) = n_0 - n$ and ξ gets constant integer values in any interval located in the resolvent sets of both H and H_0 .

Before applying this to our situation, we assume the existence of $c \in \mathbb{R}$ such that $H_0 + c1$ and $H + c1$ are positive definite, and

$$(R(H))^n(z) - (R(H_0))^n(z) \in \mathcal{L}^1(\mathcal{H}) \quad (12)$$

for some $z \leq -c$ and $n \in \mathbb{N}$.

Now, define ξ by

$$\xi(\lambda) := -\xi((\lambda + c)^{-n}; (H + c1)^{-n}, (H_0 + c1)^{-n}) \quad (13)$$

for $\lambda > -c$ and $\xi(\lambda) = 0$ for $\lambda \leq -c$.

Let $\mathcal{C}_{1,n}$ be the set of functions f having two locally bounded derivatives and satisfying

$$(\lambda^{n+1} f'(\lambda))'(\lambda) \underset{\lambda \rightarrow \infty}{\sim} \mathcal{O}(\lambda^{-1-\epsilon}) \text{ for some } \epsilon > 0.$$

When $f \in \mathcal{C}_{1,n}$, $f(H) - f(H_0) \in \mathcal{L}^1(\mathcal{H})$ and the trace formula (11) holds true.

Remark 2.1. For instance, f defined by $f(\lambda) = (\lambda - z)^{-n}$ for z in the resolvent set of H and H_0 is in $\mathcal{C}_{1,n}$; typically, $f_r(\lambda) = (\lambda + a)^{-r}$ is in $\mathcal{C}_{1,n}$ for $r > n$ and $a > 0$.

Moreover, $\text{Tr}(R^n(z) - R_0^n(z)) = -n \int_{-\infty}^{\infty} d\lambda \xi(\lambda) (\lambda - z)^{-(n+1)}$. Similarly, for $b < \inf \text{Spect}(H)$,

$$\begin{aligned} \text{Tr}(e^{-tH} - e^{-tH_0}) &= -t \int_{-\infty}^{\infty} d\lambda \xi(\lambda) e^{-t\lambda} \\ &= (2\pi i)^{-1} (n-1)! t^{-n+1} \int_{b-i\infty}^{b+i\infty} dz \text{Tr}(R^n(z) - R_0^n(z)) e^{-tz}. \end{aligned}$$

For any triplet of selfadjoint operators H_0, H_1, H_2 with trace-class differences, the equality $\xi(\lambda; H_2, H_0) = \xi(\lambda; H_2, H_1) + \xi(\lambda; H_1, H_0)$ is equivalent to the additivity of spectral action (4).

We shall now apply the above results to $H = \mathcal{D}_A^2$ and $H_0 = \mathcal{D}^2$. It is known that H_0 has a purely absolutely continuous spectrum \mathbb{R}^+ with infinite multiplicity for $d \geq 2$.

2.2 Trace-class resolvent perturbation and G -pseudodifferential calculus

The goal of this section is to prove the following

Theorem 2.2. *Let $H = \mathcal{D}_A^2$ and $H_0 = \mathcal{D}^2$ defined in (2), with A satisfying (3).*

If $c > 0$, then condition (12) is satisfied with $n = m(= d/2)$, so that ξ as defined in (13) exists and Eq. (11) holds true for $f \in \mathcal{C}_1 := \mathcal{C}_{1,m}$. In particular, the variation of the non-compact spectral action (4) is well-defined for any $f \in \mathcal{C}_1$ and $\Lambda \in \mathbb{R}^+$.

We start with a lemma about resolvent perturbations for abstract selfadjoint operators on a Hilbert space. We denote $R_T : c \in \mathbb{R}^+ \rightarrow (T^2 + c)^{-1} \in \mathcal{B}(\mathcal{H})$ for any selfadjoint operator T on a Hilbert space \mathcal{H} .

Lemma 2.3. *Let \mathcal{H} be a Hilbert space, P be an unbounded selfadjoint operator on \mathcal{H} and denote for any $k \in \mathbb{N}$, H_P^k the Hilbert space based on $\text{Dom } P^k$, endowed with the scalar product*

$$(\psi, \phi)_k := ((P - i)^k \psi, (P - i)^k \phi)_{\mathcal{H}} + (\psi, \phi)_{\mathcal{H}}.$$

Let $n \in \mathbb{N}$ and A be a selfadjoint bounded operator on \mathcal{H} such that A sends continuously H_P^k into itself for any $k \in \{0, \dots, n\}$.

The following holds for any $k \in \{0, \dots, n\}$:

(i) $H_{P+A}^k = H_P^k$ (with equivalent norms).

(ii) For any $c > 0$, $(R_P(c))^{-k}(R_{P+A}(c))^k$ is a bounded operator on \mathcal{H} . Moreover,

$$(R_P(c))^k - (R_{P+A}(c))^k = \sum_{j=1}^k (R_P(c))^j (AP + PA + A^2)(R_{P+A}(c))^{k+1-j}.$$

(iii) The operator $(R_P(c))^k - (R_{P+A}(c))^k$ is trace-class on \mathcal{H} if for any $j \in \{1, \dots, k\}$, the operator $(R_P(c))^j (AP + PA + A^2)(R_P(c))^{k+1-j}$ is trace-class on \mathcal{H} .

Proof. (i) Let us check by induction that $\text{Dom } P^k = \text{Dom}(P + A)^k$. It is clearly true for $k = 0, 1$. Suppose that it holds for a given $1 \leq k \leq n-1$. We see that if $\psi \in \text{Dom}(P + A)^{k+1}$ then $\psi \in \text{Dom } P^k$ is such that $(P + A)\psi \in \text{Dom } P$. But since $A \text{Dom } P^k \subseteq \text{Dom } P^k \subseteq \text{Dom } P$, we get $P\psi \in \text{Dom } P$, which implies $\psi \in \text{Dom } P^{k+1}$. Similarly, we get the other inclusion $\text{Dom } P^{k+1} \subseteq \text{Dom}(P + A)^{k+1}$.

We now look at the norm equivalence. Fix k with $0 \leq k \leq n$. We first claim that the operator $B := (P + A - i)^k (P - i)^{-k}$ is bounded on \mathcal{H} . To prove this, we take $\psi \in \text{Dom } P^k = \text{Dom}(P + A)^k$ and we expand $(P + A - i)^k \psi = (P - i)^k \psi + X\psi$ where X is a sum of terms of the form $\prod_{p=1}^k X_p$ where each X_p is either A or $P - i$, and there is a least one p such that $X_p = A$. Since A sends continuously H_P^m into itself for any $m \in \{0, \dots, n\}$, we see that X sends continuously H_P^k into H_P^1 . In particular, $X(P - i)^{-k}$ is bounded on \mathcal{H} . If $\psi \in \mathcal{H}$, $(P - i)^{-k} \psi \in \text{Dom } P^k$, and thus $B\psi = \psi + X(P - i)^{-k} \psi$, which proves the claim. The bounded inverse theorem now entails that $B^{-1} = (P - i)^k (P + A - i)^{-k}$ is also bounded on \mathcal{H} .

Denoting $\|\cdot\|_{A,k}$ (resp. $\|\cdot\|_k$) the Hilbert norm associated to H_{P+A}^k (resp. H_P^k), we see that for any $\psi \in \text{Dom } P^k = \text{Dom}(P + A)^k$,

$$\|\psi\|_{A,k}^2 \leq \max\{1, \|B\|^2\} \|\psi\|_k^2, \quad \|\psi\|_k^2 \leq \max\{1, \|B^{-1}\|^2\} \|\psi\|_{A,k}^2$$

from which the result follows.

(ii) Clearly $((P + A)^2 + c)^{-k}$ sends continuously \mathcal{H} into H_{P+A}^{2k} . Therefore, by (i), it is continuous from \mathcal{H} into H_P^{2k} . Moreover, P^{2k} sends continuously H_P^{2k} into \mathcal{H} . As a consequence, the composition $(R_P(c))^{-k}(R_{P+A}(c))^k$ is a bounded operator on \mathcal{H} .

For the second statement, we use $R'(c) = (R(c))^2$ after k derivations of the resolvent identity $R_P(c) - R_{P+A}(c) = R_P(c)(AP + PA + A^2)R_{P+A}(c)$ with respect to the parameter c .

(iii) follows directly from (ii). \square

We now recall some definitions and properties of G -pseudodifferential operators on \mathbb{R}^d [20]: let $G^{p,q}(\mathbb{R}^d)$ (resp. $OPG^{p,q}$) be the G -class of symbols (resp. pseudodifferential operators) of order (p, q) , valued in $\mathcal{M}_{2m}(\mathbb{C})$, with $d = 2m$; recall that $a \in G^{p,q}(\mathbb{R}^d)$ if and only if for any multi-index $(\alpha, \beta) \in \mathbb{N}^{2d}$,

$$\left\| \partial_\xi^\alpha \partial_x^\beta a(x, \xi) \right\| \lesssim \langle \xi \rangle^{p-|\alpha|} \langle x \rangle^{q-|\beta|}$$

where $\langle y \rangle := (1 + \|y\|^2)^{1/2}$.

We denote \mathcal{R}_ρ , where $\rho > 0$, the space of all bounded operators on $\mathcal{H} := L^2(\mathbb{R}^d, \mathbb{C}^{N^{2m}})$ that

- sends continuously $H^k := H^k(\mathbb{R}^d, \mathbb{C}^{N^{2m}})$ into itself for all $k \in \{0, \dots, m\}$.
- extends as bounded operator from $L^{2,-s}$ into $L^{2,-s+\rho}$ for any $s > 0$, where we define the set $L^{2,s} := L^{2,s}(\mathbb{R}^d, \mathbb{C}^{N^{2m}})$ as the weighted Hilbert space of functions such that $\|\langle x \rangle^s f\|_{\mathcal{H}} < \infty$.

We recall that N denotes dimension of the gauge group representation.

The interest of the space \mathcal{R}_ρ lies in the following result.

Lemma 2.4. *Let \tilde{P} be an elliptic symmetric operator in $OPG^{1,0}$ and $A \in \mathcal{R}_\rho$ be selfadjoint with $\rho > d$. Denote P the selfadjoint unbounded operator acting as \tilde{P} on \mathcal{H} with domain H^1 . Then for any $c > 0$,*

$$(R_P(c))^m - (R_{P+A}(c))^m \in \mathcal{L}^1(\mathcal{H}).$$

Proof. Note that we can suppose without loss of generality that $\rho \leq d + 1$.

Since $\tilde{P} \in OPG^{1,0}$ is elliptic, $H_P^k = H^k$ for any $k \in \mathbb{N}$. By Lemma 2.3 (iii), since it is supposed that A sends H^k into itself for any $0 \leq k \leq m$, it is sufficient to check that for any $j \in \{1, \dots, m\}$, the operator $(R_P(c))^j (AP + PA + A^2) (R_P(c))^{m+1-j}$ is trace-class on \mathcal{H} .

Fix $1 \leq j \leq m$. We will now prove that $(R_P(c))^j AP (R_P(c))^{m+1-j} \in \mathcal{L}^1(\mathcal{H})$. The proof that $(R_P(c))^j PA (R_P(c))^{m+1-j}$ and $(R_P(c))^j A^2 (R_P(c))^{m+1-j}$ are in $\mathcal{L}^1(\mathcal{H})$ is similar.

Let M_s be the pseudodifferential operator with Weyl symbol $\langle x \rangle^s$, for $s \in \mathbb{R}$. The operator M_s is elliptic invertible in $OPG^{0,s}$, with inverse M_{-s} , and sends continuously $L^{2,s'}$ into $L^{2,s'-s}$, for any $s' \in \mathbb{R}$. By hypothesis A can be extended so that $M_{2j} A M_{\rho-2j}$ is bounded on \mathcal{H} . In other words, there exists a bounded operator $B \in \mathcal{B}(\mathcal{H})$ such that $A = M_{-2j} B M_{-\rho+2j}$.

Note also that $(R_P(c))^j$ extends as pseudodifferential operator in $OPG^{-2j,0}$ since \tilde{P} is elliptic in $OPG^{1,0}$.

Fix $0 < \varepsilon < \min\{\frac{1}{2}(\rho/d - 1), 2j/d, (\rho - 2j)/d\}$. It is known [6, 7] that if $T \in OPG^{p,q}$, $p < -d/r$, $q < -d/r$, then $T \in \mathcal{L}^r(\mathcal{H})$. Thus, it appears that $(R_P(c))^j M_{-2j} \in \mathcal{L}^{p_1}(\mathcal{H})$ where $p_1^{-1} = 2j/d - \varepsilon$ and $M_{-\rho+2j} P (R_P(c))^{m+1-j} \in \mathcal{L}^{p_2}(\mathcal{H})$ where $p_2^{-1} = (\rho - 2j)/d - \varepsilon$. As a consequence,

$$(R_P(c))^j AP (R_P(c))^{m+1-j} = (R_P(c))^j M_{-2j} B M_{-\rho+2j} P (R_P(c))^{m+1-j} \in \mathcal{L}^{(p_1^{-1} + p_2^{-1})^{-1}}(\mathcal{H})$$

and the result now follows from the fact that $p_1^{-1} + p_2^{-1} = \rho/d - 2\varepsilon \geq 1$. \square

Proof of Theorem 2.2. The condition (3) implies that the multiplication operator A belongs to \mathcal{R}_ρ . Thus, Lemma 2.4 applied to $\tilde{P} = \mathcal{D}$ yields directly the result. \square

Remark 2.5. *When A is a multiplication operator, the condition $A \in \mathcal{R}_\rho$ implies some constraints on the derivatives of A . To remove these constraints, a technique based on a commutation between R_{P+A} and M_s was used in [28]. On the other hand, our class \mathcal{R}_ρ contains operators that are not necessarily multiplication by functions. For instance $M_{-\rho} P \in \mathcal{R}_\rho$ when P is a pseudodifferential operator on \mathbb{R}^d with symbol in the Hörmander class $S_{0,0}^0$.*

Remark 2.6. In the stronger case where A is supposed to be a pseudodifferential operator in $OPG^{0,-\rho}$ where $\rho > d$, then the trace-class property $(R_P(c))^m - (R_{P+A}(c))^m \in \mathcal{L}^1(\mathcal{H})$ follows more directly and there is a integral formula for the value of trace of the resolvent perturbation $(R_P(c))^m - (R_{P+A}(c))^m$.

Indeed, in this case, $P+A$ is elliptic in $OPG^{1,0}$. By ellipticity and spectral invariance, R_A and R are in $OPG^{-2,0}$. Since $\rho > 0$, $AP + PA + A^2 \in OPG^{1,-\rho}$ implies that $R_P^m - R_{P+A}^m \in OPG^{-2n-1,-\rho}$. The result now follows from the following property (see for instance [20, Theorem 4.4.21] for more details and greater generality): if $T \in OPG^{p,q}$ with $p < -d$, $q < -d$, then T is a trace-class operator on $L^2(\mathbb{R}^d, \mathbb{C}^{N^{2m}})$, and moreover $\text{Tr}(T) = \int_{\mathbb{R}^{2d}} \text{tr}_{\mathbb{C}^{N^{2m}}}(T_w)$ where T_w is the Weyl symbol of T which belongs to $G^{-d,-d}(\mathbb{R}^d) \subseteq L^1(\mathbb{R}^{2d}, \mathbb{C}^{N^{2m}})$.

3 Spectral action to the second order of field strength

3.1 Heat kernel to the second order of field strength

Despite the fact that in [3], M is only supposed to be asymptotically flat, we assume here that M is flat. Equation (2) yields

$$\mathcal{D}_A^2 = -(\nabla^2 + E), \quad \nabla^2 := g^{\mu\nu} \nabla_\mu \nabla_\nu, \quad E := \frac{1}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu}.$$

We first compute the trace of the heat kernel as a function of F . In [3], Barvinsky and Vilkovisky have introduced a covariant perturbation theory, and the use of [3, eq. (2.1)] gives, for any $s > 0$, small or large,

$$\tilde{K}(s, \mathcal{D}_A^2)(F) = \frac{1}{(4\pi s)^m} \int_M d^{2m}x \text{tr} \left[sE + s^2 E \frac{1}{2} h(-s\partial^2) E + s^2 F_{\mu\nu} q(-s\partial^2) F^{\mu\nu} \right] + \mathcal{O}(F^3)(s) \quad (14)$$

where the trace is both on the gauge and spinor indices and

$$q(z) := -\frac{1}{2} \frac{h(z)-1}{z}, \quad h(z) := \int_0^1 d\alpha e^{-\alpha(1-\alpha)z}. \quad (15)$$

The function h appears here since it corresponds to an expansion process:

$$h(z) = \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \delta(1 - \alpha_1 - \alpha_2) e^{-\alpha_1 \alpha_2 z}.$$

The main interest of formula (14) is that it is valid for all positive values of s , and not only when $s \rightarrow 0$. Thus if

$$\kappa(z) := 2^{m-2} (-h(z) + 4q(z)), \quad (16)$$

we get, with $\text{tr}([\gamma^\mu, \gamma^\nu][\gamma^\rho, \gamma^\sigma]) = 2^{m+2}(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma})$,

$$\text{Tr}(\exp(-s\mathcal{D}_A^2) - \exp(-s\mathcal{D}^2))(F) = \frac{1}{(4\pi)^m} s^{-m+2} \int_M d^{2m}x \text{tr} [F_{\mu\nu} \kappa(-s\partial^2) F^{\mu\nu}] + \mathcal{O}(F^3)(s) \quad (17)$$

where the trace is now over gauge indices only. Note that

$$h(z) = 2 \sum_{n=0}^\infty \frac{z^n}{n!} \left(\frac{d}{dz}\right)^{2n} \int_0^{1/2} d\alpha e^{-\alpha z} \underset{z \rightarrow \infty}{\sim} 2 \sum_{j=0}^\infty \frac{z^{-1-j}(2j)!}{j!}. \quad (18)$$

So we obtain the asymptotics of $\kappa(z) = 2^m h'(z)$:

$$\kappa(z) \underset{z \rightarrow \infty}{\sim} -2^{m+1} z^{-2}. \quad (19)$$

3.2 Assumptions on the function f

Let \mathcal{C}_2 be the class of functions f such that

$$\begin{aligned} f : [0, \infty[\rightarrow \mathbb{R}^+ \text{ is piecewise continuous and} \\ f(x) \underset{x \rightarrow \infty}{\sim} \mathcal{O}(f_\epsilon(x)) \text{ with } f_\epsilon(x) := (x + \epsilon)^{-m-\epsilon} \text{ for some } \epsilon > 0 \end{aligned} \quad (20)$$

where we take care of the eventually (finite dimensional) kernel of \mathcal{D}^2 via $x + \epsilon$.

Let \mathcal{C}_3 be the class of functions f such that

$$\begin{aligned} f \text{ is a function (not a distribution) such that exists a function } \varphi \text{ satisfying} \\ f = L\varphi \text{ and} \\ \text{either } \varphi(t) \text{ is a function and } t^{-k} \varphi(t) \in \mathcal{L}^1([0, \infty[) \text{ for } k \in \{1, \dots, m\} \\ \text{or } \varphi \text{ is a finite sum of } t^p \delta^{(n)}(t - a) \text{ for arbitrary } n, p \in \mathbb{N}, a \in]0, \infty[. \end{aligned} \quad (21)$$

The set $\mathcal{C}_1 \cap \mathcal{C}_2$ contains for instance $(a + x)^n e^{-bx}$ for $a, b \geq 0, n \in \mathbb{N}$, or completely monotonic functions (see Definition 3.2) which fall off like in (20). We can enlarge this set with all finite positive linear combinations of functions satisfying (21).

3.3 Calculation of the spectral action

We first assume that $f = L\varphi$ is a Laplace transform of a function φ , as in (5), so the spectral action is given by (6).

The expansion (9) starts with $k = 1$. However, the same universal function f should describe the spectral action in the presence of metric fluctuations when the heat kernel coefficient a_0 does not vanish. For this reason, φ_0 must be finite,

$$c_f := \varphi_0 = \int_0^\infty dt t^{-m} \varphi(f)(t) < \infty. \quad (22)$$

With the definition

$$w_\Lambda(z) := \int_0^\infty dt t^{-m+2} \kappa(t\Lambda^{-2} z) \varphi(t), \quad (23)$$

we obtain

$$\int_M d^{2m}x \operatorname{tr}[F^{\mu\nu} w_\Lambda(-\partial^2) F_{\mu\nu}] = \int d^{2m}p \operatorname{tr}[\hat{F}^{\mu\nu}(-p) w_\Lambda(p^2) \hat{F}_{\mu\nu}(p)] \quad (24)$$

where \hat{F} is the Fourier transform of F . At this point, it is important to quote that, by the analysis in [3], at all order in the curvature, the large s behavior of $\operatorname{Tr}(\exp(-s\mathcal{D}_A^2))(F)$ in (17) is in s^{-m+2} . In particular, in dimension 4, it stabilizes.

So the F -dependence of the spectral action (assuming commutation of integrals) is

$$S(\mathcal{D}_A, \Lambda, f)(F) = \frac{\Lambda^{d-4}}{(4\pi)^m} \int_M d^{2m}p \operatorname{tr}[\hat{F}^{\mu\nu}(-p) w_\Lambda(p^2) \hat{F}_{\mu\nu}(p)] + \mathcal{O}(F^3). \quad (25)$$

We emphasize now that it is not necessary to assume that the function f is a Laplace transform of a function φ which could be a distribution. This is a point where we can enjoy the beauty of the spectral action. Here, this assumption is an artifact due to the use of the heat kernel in formula (17).

For all $k \in \{1, \dots, m\}$, denote by $f^{[k]}(x) := -\int_x^\infty dy f^{[k-1]}(y)$ be a k -th primitive of f (so we have $(f^{[k]})^{(k)} = f$).

Lemma 3.1. Assume $f \in \mathcal{C}_2$. Then, for $k \in \{1, \dots, m\}$, $(-1)^k f^{[k]}$ is a bounded positive function in $C^k([0, \infty[)$ such that $(-1)^k f^{[k]} = \mathcal{O}((-1)^k f_\epsilon^{[k]})$. Moreover

$$\int_0^\infty ds \int_s^\infty ds_1 \int_{s_1}^\infty ds_2 \cdots \int_{s_{k-2}}^\infty ds_{k-1} f(s_{k-1}) = (-1)^k f^{[k]}(0) \geq 0.$$

Proof. Positivity and differentiability of $-f^{[1]}$ are clear. Let x_0 be such that $f(x) \leq c f_\epsilon(x)$, $\forall x > x_0$. Then, $0 \leq -f^{[1]}(x) \leq c \int_x^\infty dy f_\epsilon^{[1]}(y) \leq -c f_\epsilon^{[1]}(x)$ and $-f^{[1]} = \mathcal{O}(-f_\epsilon^{[1]})$. An induction yields $0 \leq (-1)^k f^{[k]} = \mathcal{O}((-1)^k f_\epsilon^{[k]})$. The integral identity follows from the fact that $f^{[k]}(x) \rightarrow 0$ when $x \rightarrow \infty$. \square

Since we want to make a connection to (22) and (23), we are obliged to suppose that f is a Laplace transform but eventually of a distribution since, typically, we want to allow $f(x)$ to be e^{-x} .

The next lemma shows $f \in \mathcal{C}_2$ should be sufficient to get Theorem 3.4 at the price to redo all the covariant perturbation theory results of [3], something that we do not do here. So we will assume that f is a Laplace transform of a certain class of distributions defined on purpose.

It is worthwhile to quote at this point that there is a nice class of functions satisfying (20) (if one adjusts the tail of the function), namely the class of completely monotonic functions:

Definition 3.2. A function $f :]0, \infty[\rightarrow \mathbb{R}$ is completely monotonic (c. m.) if $f^{(n)}(x)$ exists for all $n \in \mathbb{N}$ and $(-1)^n f^{(n)}(x) \geq 0$ for all $x > 0$.

The limit $f^{(n)}(0) = \lim_{x \rightarrow 0^+} f^{(n)}(x)$ exists, finite or infinite. By Bernstein's theorem, a necessary and sufficient condition that f should be c. m. in $]0, \infty[$ is that $f(x) = \int_0^\infty e^{-xt} dg(t)$ where $g(t)$ is a bounded non-decreasing function and that the integral converges for $x \in]0, \infty[$ (see [26, p. 161]). If $0 \neq f$ is c. m., then $f(x)$ cannot vanish, and when $f^{(n)}(x_0) = 0$ for $x_0 > 0$, then $f^{(m)}(x_0) = 0$ for all $m > n$.

Examples of c. m. functions: $e^{-\alpha x}$, $(\alpha + \beta x)^\gamma$, $\log(a + \frac{b}{x})$ for $\alpha, \beta, \gamma \geq 0$, $a \geq 1, b > 0$. This class is quite stable: if f and g are c. m. then, $f^{(2n)}(x)$, $-f^{(2n+1)}(x)$, $af(x) + bg(x)$ (for $a, b \geq 0$), $f(x)g(x)$, $e^{f(x)}$ are c. m. In particular e^{ax^b} , $(1+x)^{a/x}$, $(\alpha + \frac{\beta}{x})^a$ are c. m. for $a \geq 0, b \leq 0, \alpha \geq 1, \beta > 0$.

Any Laplace transform $L(\varphi)(x) = \int_0^\infty dt e^{-xt} \varphi(t)$ of a non-negative function φ is c. m. (if integral converges!). Any c. m. function f is positive and Laplace transform of a positive function φ . Note that f_ϵ is c. m. and in κ defined in (16), the key function h of (15) is also c. m. and so is $-\kappa(x) = -2^m h'(x)$.

To allow commutation of integrals in the proof of next lemma, we will use the hypothesis defined in Section 3.2.

Lemma 3.3. Assume that f satisfies (20) and (21). Then

$$c_f = (-1)^m f^{[m]}(0), \tag{26}$$

$$w_\Lambda(p^2) = (-1)^{m-1} 2^{m-2} \int_0^1 d\alpha \left[f^{[m-2]}(\alpha(1-\alpha)\Lambda^{-2}p^2) - \frac{2\Lambda^2}{p^2} \int_0^{\alpha(1-\alpha)\Lambda^{-2}p^2} ds_1 f^{[m-2]}(s_1) \right]. \tag{27}$$

Proof. The first equation follows from $c_f = L(t^{-m}\varphi(t))(0)$ and a property of Laplace transform: for $k \in \mathbb{N}^*$, $L(t^{-k}\varphi(t))(s) = \int_s^\infty ds_1 \cdots \int_{s_{k-1}}^\infty ds_k f(s_k)$, so that we can use Lemma 3.1. We apply again this formula to $\kappa(z) = 2^{m-2}(-h(z) - 2 \frac{h(z)-1}{z})$ in the function $w_\Lambda(\Lambda^2 x) = \int_0^\infty dt \kappa(tx) \varphi(t) t^{-(m-2)}$:

for instance,

$$\begin{aligned}
\int_0^\infty dt h(tx) \varphi(t) t^{-(m-2)} &= \int_0^\infty dt \int_0^1 d\alpha e^{-\alpha(1-\alpha)tx} \varphi(t) t^{-(m-2)} \\
&= \int_0^1 d\alpha \int_0^\infty dt e^{-(\alpha(1-\alpha)x)t} \varphi(t) t^{-(m-2)} \\
&= \int_0^1 d\alpha L(\varphi(t) t^{-(m-2)})(\alpha(1-\alpha)x) \\
&= \int_0^1 d\alpha \int_{\alpha(1-\alpha)x}^\infty ds_1 \int_{s_1}^\infty ds_2 \cdots \int_{s_{m-3}}^\infty ds_{m-2} f(s_{m-2}) \\
&= \int_0^1 d\alpha (-1)^{m-2} f^{[m-2]}(\alpha(1-\alpha)x). \tag{28}
\end{aligned}$$

The commutation of integrals in the second equality follows from (21). To complete the proof, we proceed similarly for

$$\begin{aligned}
\int_0^\infty dt \frac{h(tx)-1}{tx} \frac{\varphi(f)(t)}{t^{m-2}} &= \frac{1}{x} \int_0^\infty dt h(tx) \frac{\varphi(f)(t)}{t^{m-1}} - \frac{1}{x} \int_0^\infty dt \frac{\varphi(f)(t)}{t^{m-1}} \\
&= \frac{1}{x} \int_0^1 d\alpha [L(\varphi(t)t^{1-m-1})(\alpha(1-\alpha)x) - L(\varphi(t)t^{1-m})(0)] \\
&= -\frac{1}{x} \int_0^1 d\alpha \int_0^{\alpha(1-\alpha)x} ds_1 \int_{s_1}^\infty ds_2 \cdots \int_{s_{m-2}}^\infty ds_{m-1} f(s_{m-1}) \\
&= \frac{(-1)^{m-1}}{x} \int_0^1 d\alpha \int_0^{\alpha(1-\alpha)x} ds_1 f^{[m-2]}(s_1). \quad \square
\end{aligned}$$

For instance, in dimension $d = 4$, formulae (26) and (27) read

$$\begin{aligned}
c_f &= \int_0^\infty ds \int_s^\infty ds_1 f(s_1), \\
w_\Lambda(p^2) &= - \int_0^1 d\alpha \left[f(\alpha(1-\alpha)\Lambda^{-2}p^2) - \frac{2\Lambda^2}{p^2} \int_0^{\alpha(1-\alpha)\Lambda^{-2}p^2} ds_1 f(s_1) \right]. \tag{29}
\end{aligned}$$

Finally, we get the following

Theorem 3.4. *Assume that $f \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3$. Then, up to the second order in F , the variation of the spectral action of above spectral triple is given by (25) where c_f and w are defined by (26) and (27). Moreover the asymptotics of w is controlled by*

$$w_\Lambda(p^2) \underset{p^2 \rightarrow \infty}{\sim} -2^{m+1} c_f \Lambda^4 p^{-4} + o(p^{-4}). \tag{30}$$

Proof. It remains to prove the asymptotics. Since $\kappa(x) = 2^m \frac{d}{dx} h(x)$, $\kappa(t\Lambda^{-2}z) = \frac{2^m \Lambda^2}{t} \frac{d}{dz} h(t\Lambda^{-2}z)$, with $z := p^2$,

$$\begin{aligned}
w_\Lambda(z) &= 2^m \Lambda^2 \frac{d}{dz} \int_0^\infty dt t^{-m+1} h(t\Lambda^{-2}z) \varphi(t) \\
&= 2^m \Lambda^2 \frac{d}{dz} \int_0^1 d\alpha (-1)^{m-1} f^{[m-1]}(\alpha(1-\alpha)z\Lambda^{-2}) \tag{31}
\end{aligned}$$

by (28). Moreover

$$\begin{aligned}
\int_0^1 d\alpha (-1)^{m-1} f^{[m-1]}(\alpha(1-\alpha)z\Lambda^{-2}) &\underset{z \rightarrow \infty}{\sim} (-1)^{m-1} 2 \int_0^{1/2} d\alpha f^{[m-1]}(\alpha z \Lambda^{-2}) \\
&\underset{z \rightarrow \infty}{\sim} \frac{(-1)^{m-1} 2\Lambda^2}{z} \int_0^\infty dy f^{[m-1]}(y) \\
&= \frac{-2(-1)^{m-1} \Lambda^2}{z} f^{[m]}(0) = \frac{2\Lambda^2}{z} c_f.
\end{aligned}$$

Thus

$$w_\Lambda(z) \underset{z \rightarrow \infty}{\sim} 2^{m+1} \Lambda^4 c_f \frac{4}{dz} \left(\frac{1}{z} \right). \quad \square$$

Remark 3.5. The coefficient $\Lambda^4 c_f$ is the same one which appears in front of the a_0 term in the weak-field asymptotics of the spectral action. Although in our case, this coefficient is canceled due to the subtraction of the free heat kernel, cf. Eq. (9), it contributes to the cosmological constant term for more general spectral triples allowing for metric perturbations [14]. This is a very remarkable fact which shows an intimate relation between leading terms in the low and high momenta asymptotics of the spectral action. A positive value for c_f is required to reproduce the Standard Model from noncommutative geometry [12, 14] (see also discussion in Sec. 4.1 below).

Remark 3.6. While the spectral action is non-local (a price to pay for its extremely simple definition), it becomes local after some asymptotic expansion. But the covariant perturbation method used from [3] is a way to control non-localities.

Remark 3.7. To get the theorem, a spin structure is not necessary on M : any second order operator of the Laplace type with values on a vector bundle V over M (see [25, eq. (2.1)]) can replace \mathcal{D}^2 .

Remark 3.8. By Theorem 3.4, the function $w_\Lambda(p^2)$ decays at large p at least as $1/p^4$. This implies that the propagator for the Yang-Mills field grows at large momenta, and the Yang-Mills spectral action is not super-renormalizable in contrast to the restricted expansion of the spectral action studied in [24].

3.4 An example

Assume that $d = 4$, and as a function f , let us take the function $f_r(z) := (z + a)^{-r}$ having a power-low decay at infinity where $0 \neq a \in \mathbb{R}^+$ and $r > 2$, so $f \in \mathcal{C}_1$. To estimate the asymptotic behavior of spectral action it is most convenient to use (31). Since the first primitive of f is $f_r^{[1]} = (1 - r)^{-1}(z + a)^{-r+1}$, we have

$$\begin{aligned} w_{\Lambda,r}(z) &= 4\Lambda^2 \frac{d}{dz} \int_0^1 d\alpha \frac{-1}{1-r} [\alpha(1-\alpha)z\Lambda^{-2} + a]^{-r+1} = \frac{8}{r-1} \frac{d}{dy} \int_0^{1/2} d\alpha [\alpha(1-\alpha)y + a]^{-r+1} \\ &= \frac{8}{r-1} \frac{d}{dy} y^{-r+1} \int_0^{1/2} d\alpha \left[\alpha(1-\alpha) + \frac{a}{y} \right]^{-r+1} \end{aligned}$$

with $y = z\Lambda^{-2} = p^2\Lambda^{-2}$. This expression is valid for arbitrary p^2 .

To evaluate the asymptotic behavior at $p^2 \rightarrow \infty$ (or, at $y \rightarrow \infty$), one represents the integrand $(a/y) + \alpha(1-\alpha) = (a/y) + \alpha - \alpha^2$ and expands in α^2 :

$$w_{\Lambda,r}(p^2) = \frac{8}{r-1} \frac{d}{dy} y^{-r+1} \int_0^{1/2} d\alpha \left[(\alpha + (a/y))^{-r+1} - (1-r)\alpha^2(\alpha + (a/y))^{-r} + \dots \right]$$

yielding for the leading asymptotics

$$w_{\Lambda,r}(p^2) = -\frac{8}{(4\pi)^2 a^{r-2}(r-1)(r-2)} \frac{\Lambda^4}{p^4} + o(p^{-4})$$

as claimed in (30) since $c_f = \frac{1}{a^{r-2}(r-1)(r-2)}$.

It is interesting to estimate the next-to-leading term in the expansion, which behaves as

$$\begin{array}{ll} p^{-6} & \text{for } r > 3, \\ p^{-6} \ln p & \text{for } r = 3, \\ p^{-2r} & \text{for } 2 < r < 3. \end{array}$$

This example show that the next to leading term in the asymptotics is not always p^{-6} as one might think.

4 Relaxing assumptions on the function f

In the previous section, we used different sets of assumptions on the function f which defines the spectral action. The first set \mathcal{C}_1 refers to the role of f as a regulator of the trace while \mathcal{C}_2 allows to take primitives of f . It is natural to assume that f is positive, and it is necessary to assume that it falls off sufficiently fast. The third set \mathcal{C}_3 is of a technical nature: we need f being a Laplace transform of a function φ to use the heat kernel methods. The form of φ was restricted to enable us to commute integrals in the proof of Theorem 3.1.

One can ask whether there is a natural family of functions satisfying all assumptions which we have made. Such a family consists of the so-called completely monotonic functions and is described in the Definition 3.2. These functions seem to be the best candidates for the definition of spectral action.

Another question, which shall be address below, is whether some of the assumptions can be weakened.

4.1 Lifting the positivity assumption

For a positive function f , the k -th primitive $f^{[k]}$ is positive for k even and negative for k odd. Then, according to (26) the coefficient c_f is strictly positive. The leading $1/p^4$ term in the expansion (30) is then non-zero. However, lifting the positivity assumption on f , one gets a possibility to engineer a variation of spectral action such that $w_\Lambda(p^2)$ falls off at large p as p^{-2k} for any given integer $k > 2$. Indeed, consider f being a finite sum

$$f(z) = \sum_n f_n(z), \quad f_n(z) = c_n \exp(-z/b_n)$$

with $b_n > 0$ and where some of c_n are allowed to take negative values. Recall that $c_n^{-1} f_n \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3$.

The variation of spectral action is then given by a sum of the heat kernels

$$\mathcal{S}(\mathcal{D}_A, \Lambda, f) = \sum_n c_n \tilde{K}(b_n^{-1} \Lambda^{-2}, \mathcal{D}_A^2), \quad (32)$$

and the function $w_\Lambda(p^2)$ can be calculated directly without using the Laplace transform:

$$w_\Lambda(p^2) = 4\Lambda^2 \frac{d}{d(p^2)} \sum_n c_n b_n h(p^2 \Lambda^{-2} b_n^{-1}).$$

Here, for simplicity, we restrict ourselves to four dimensions, so $m = 2$.

Using (18), the large p -asymptotic expansion for $w_\Lambda(p^2)$ reads

$$w_\Lambda(p^2) \underset{p \rightarrow \infty}{\sim} -8 \sum_{j=0}^{\infty} (\Lambda/p)^{2(j+2)} \frac{(j+1)(2j)!}{j!} \sum_n c_n b_n^{2+j}. \quad (33)$$

By comparing (32) with (9), we see that in the weak field asymptotics of the action, the heat kernel coefficients a_{2k} are multiplied by a sum $\sum c_n b_n^{2-k}$ having the same structure as the sums appearing in the large p -asymptotics (33) but with different powers of b_n . The only exception is $\sum c_n b_n^2$ which multiplies the leading p^{-4} term in (33) and the coefficient a_0 . (We remind, that a_0 reappears in the asymptotics of the spectral action if we allow for metric perturbations.) Herewith we reconfirm the statement made in Remark 3.5, but also observe that no further relations exist between the weak-field and large momentum asymptotics of the spectral action.

If one is allowed to choose freely the coefficients b_n and c_n and to take negative c_n , it is possible to cancel any finite number of terms in (33).

4.2 Step-function regularization

The Laplace transformation is smoothing: if f is a Laplace transform of φ , then f has to be at least continuous. However, the assumptions (20) remain valid also for piecewise continuous functions, and the right hand sides of the non-asymptotic (27) and asymptotic (30) formulae exist for such functions. It is therefore interesting to make a calculation for a piecewise continuous function f without using the Laplace transformation. Such case has been investigated in [15]: using the spectral density of the operator \mathcal{D} , a proof that the asymptotics of spectral action makes sense in the Cesàro sense, has been proposed when M is compact.

Following [1], let us take f being a step-function and use the following integral representation.

$$f(z) = \theta(1 - z) = \lim_{\epsilon \rightarrow +0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\beta \frac{e^{i(1-z)\beta}}{\beta - i\epsilon},$$

so that

$$\text{Tr} (f(\mathcal{D}_A^2/\Lambda^2) - f(\mathcal{D}^2/\Lambda^2)) = \lim_{\epsilon \rightarrow +0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\beta \frac{e^{i\beta}}{\beta - i\epsilon} \text{Tr} \left[\exp\left(-\frac{i\beta}{\Lambda^2} \mathcal{D}_A^2\right) - \exp\left(-\frac{i\beta}{\Lambda^2} \mathcal{D}^2\right) \right].$$

The coefficient in front of \mathcal{D}_A^2 above is imaginary, so that we are dealing with the Schrödinger kernel rather than with the heat kernel. In principle, one should study anew the expansion of this kernel in curvatures. We shall not go too deep into this problem. It is enough to mention that (i) the heat kernel for the free operator \mathcal{D}^2 becomes the Schrödinger kernel for the same operator after analytic continuation $t \rightarrow it$, and (ii) the Duhamel expansion, which may be used to derive the formulae like (14) (see, e.g. [25, Sec. 8.2]) also survives the continuation to imaginary t .

In four dimensions, $m = 2$, we therefore write

$$\text{Tr} \left[\exp\left(-\frac{i\beta}{\Lambda^2} \mathcal{D}_A^2\right) - \exp\left(-\frac{i\beta}{\Lambda^2} \mathcal{D}^2\right) \right] \sim \frac{1}{(4\pi)^2} \int d^4 p \text{tr} [\hat{F}^{\mu\nu}(-p) \kappa(i\beta p^2 \Lambda^{-2}) \hat{F}_{\mu\nu}(p)]$$

to the second order of the field strength F . The formula (25) remains valid with $w_\Lambda(p^2)$ given by

$$w_\Lambda(p^2) = \lim_{\epsilon \rightarrow +0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\beta \frac{e^{i\beta}}{\beta - i\epsilon} \kappa(i\beta p^2 \Lambda^{-2})$$

According to (16), the function $\kappa(z)$ contains two terms. Let us denote by $w_{1,\Lambda}$ (resp., $w_{2,\Lambda}$) the contribution from $-h(z)$ (respectively, from $4q(z)$).

$$\begin{aligned} w_{1,\Lambda} &= - \lim_{\epsilon \rightarrow +0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\beta \frac{e^{i\beta}}{\beta - i\epsilon} h(i\beta p^2 \Lambda^{-2}) \\ &= - \lim_{\epsilon \rightarrow +0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\beta \int_0^1 d\alpha \frac{e^{i\beta}}{\beta - i\epsilon} \exp(-i\beta \alpha(1 - \alpha)p^2 \Lambda^{-2}) = - \int_0^1 d\alpha \theta_\alpha, \end{aligned}$$

where we introduced a short-hand notation

$$\theta_\alpha := \theta(1 - \alpha(1 - \alpha)p^2 \Lambda^{-2}).$$

To evaluate $w_{2,\Lambda}$, we first observe that $q(z)$ does not have a pole at $z = 0$. Therefore, one can deform the contour of integration by replacing the interval $[-\epsilon/2, \epsilon/2]$ by a semicircle in the upper half plane centered at $\beta = 0$ of a radius $\epsilon/2$. We denote the deformed contour by C . Then we have

$$\begin{aligned} w_{2,\Lambda} &= \lim_{\epsilon \rightarrow +0} \frac{1}{2\pi i} \int_C d\beta \frac{e^{i\beta}}{\beta - i\epsilon} \frac{2(1 - h(i\beta p^2 \Lambda^{-2}))}{i\beta p^2 \Lambda^{-2}} \\ &= \lim_{\epsilon \rightarrow +0} \frac{1}{2\pi i} \int_C d\beta \frac{e^{i\beta}}{\beta - i\epsilon} \int_0^1 d\alpha \frac{2}{i\beta p^2 \Lambda^{-2}} \left[1 - \exp(-i\beta \alpha(1 - \alpha)p^2 \Lambda^{-2}) \right]. \end{aligned}$$

Next, we integrate over β . In the integral of the first (constant) term in the square brackets, the contour should be closed in the upper half plane, so that the pole at $\beta = i\epsilon$ contributes. To integrate the second term in square brackets, we close the contour upwards for positive arguments of θ_α , obtaining a contribution proportional to the residue at $\beta = i\epsilon$. For negative arguments of θ_α , the contour has to be closed in the lower half-plane, giving rise to a contribution proportional to $(1 - \theta_\alpha)$ and to a residue at $\beta = 0$. Collecting all together, we obtain

$$w_{2,\Lambda} = \int_0^1 d\alpha \lim_{\epsilon \rightarrow +0} \left(-\frac{2e^{-\epsilon}}{\epsilon p^2 \Lambda^{-2}} + \theta_\alpha \frac{2e^{-\epsilon}}{\epsilon p^2 \Lambda^{-2}} \exp(\epsilon \alpha (1 - \alpha) p^2 \Lambda^{-2}) - (1 - \theta_\alpha) \frac{-2}{\epsilon p^2 \Lambda^{-2}} \right).$$

In the limit $\epsilon \rightarrow 0$ we have

$$w_{2,\Lambda} = \int_0^1 d\alpha [2\alpha(1 - \alpha)\theta_\alpha + 2\Lambda^2 p^{-2}(1 - \theta_\alpha)].$$

One can check that $w_\Lambda = w_{1,\Lambda} + w_{2,\Lambda}$ agrees with (29). The integration over α can be performed giving

$$\begin{aligned} 2|p| < \Lambda : \quad w_\Lambda(p^2) &= -\frac{2}{3} \\ 2|p| \geq \Lambda : \quad w_\Lambda(p^2) &= 2[-\alpha_c + \frac{2\Lambda^2}{p^2}(\frac{1}{2} - \alpha_c) + \alpha_c^2 - \frac{2}{3}\alpha_c^3], \end{aligned}$$

where $\alpha_c := \frac{1}{2}(1 - \sqrt{1 - \frac{4\Lambda^2}{p^2}})$.

Also the large p asymptotic behavior $w_\Lambda(p^2) \sim -4\Lambda^4/p^4 + \mathcal{O}(1/p^6)$ agrees with (30) although the step-function does not satisfy assumptions of Theorem 3.4.

We conclude that the Laplace transform is not strictly necessary. We can even conjecture that the formulae (27) and (30) are valid for arbitrary piecewise continuous functions f satisfying the fall-off condition (20).

5 Higher terms in F

Since the heat kernel (14) is an invariant functional in the background field, it is expandable in the basis of invariants of any order. This has been explicitly computed in [4, 5] up to order 3 in the curvature F :

$$\begin{aligned} \tilde{K}(s, \mathcal{D}_A^2)(F) &= \frac{1}{(4\pi s)^m} \int_M d^{2m}x \operatorname{tr} \left\{ sE + s^2 E \frac{1}{2} h(-s\partial^2) E + s^2 F_{\mu\nu} q(-s\partial^2) F^{\mu\nu} \right. \\ &\quad + s^3 \sum_{i=1}^{11} \kappa_i(-s\partial_1^2, -s\partial_2^2, -s\partial_3^2)(R_1 R_2 R_3)(i) \\ &\quad + s^4 \sum_{i=12}^{25} \kappa_i(-s\partial_1^2, -s\partial_2^2, -s\partial_3^2)(R_1 R_2 R_3)(i) \\ &\quad + s^5 \sum_{i=26}^{28} \kappa_i(-s\partial_1^2, -s\partial_2^2, -s\partial_3^2)(R_1 R_2 R_3)(i) \\ &\quad \left. + s^6 \kappa_{29}(-s\partial_1^2, -s\partial_2^2, -s\partial_3^2)(R_1 R_2 R_3)(29) + \mathcal{O}(F^4)(s) \right\} \end{aligned} \quad (34)$$

where the list of functions κ_i is known: they are constructed like previous $\kappa_0(x) = 2^m h'(x)$ through the function

$$h(x_1, x_2, x_3) := \int_{(R^+)^3} d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) e^{-\alpha_1 \alpha_2 x_3 - \alpha_2 \alpha_3 x_1 - \alpha_1 \alpha_3 x_2}.$$

The formal expression $R_1 R_2 R_3$ means one of the cross-terms in F and other curvatures that can appear in the computation.

Exactly the same method as before can be applied. While $w_\Lambda(p^2)$ defines behavior of the propagator and partially of the A^3 and A^4 vertexes, the action (34) will define higher vertex functions.

6 Conclusions

In this paper we considered the variation of spectral action for a commutative spectral triple perturbed by a gauge potential. We calculated this variation to the second order in field strength. Our main results are the remarkably simple formula (27) for the action and its universal asymptotics, Theorem 3.4.

Although we used the Laplace transform at intermediate steps, it seems to be unnecessary, as suggests the example of step function regularization. Anyway, it would be interesting to obtain the results without relying on the Laplace transform. Other open directions for further research include extensions to odd dimensions and to higher orders in the field strength.

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